Categorification of perfect matchings

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What is “categorification” here?

Enriching/explaining combinatorics by representation theory

... for example: replacing cluster algebras by cluster categories

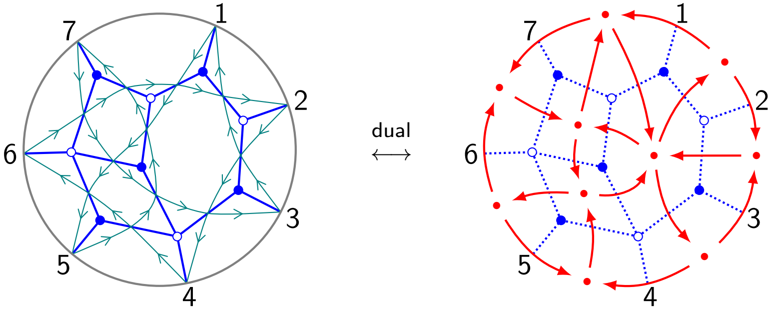
... but not: higher representation theory [Khovanov, Rouqier, ...]

(replacing vector spaces & linear maps by categories & functors)

... in this talk: interpreting *dimer models and perfect matchings*

as *quivers with faces and their representations*

Context: consistent dimer models on a disc



planar bipartite graph G

$+$ Postnikov conditions

on strands (zig-zag paths)

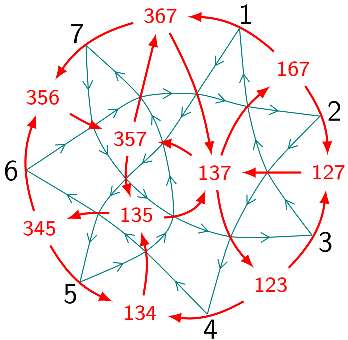
quiver with faces,

i.e. oriented cycles,

$Q = (Q_0, Q_1, Q_2)$

Combinatorics: left source labelling of quiver vertices

[Postnikov, Scott] Label each quiver vertex $i \in Q_0$ by the *sources* of the strands for which i is on the *left* of the strand.



Ex: Labels $\mathtt{J}_i$ are in $\binom{[n]}{k} = \{k\text{-subsets of } \{1, \dots, n\}\}$, for fixed k, the average increment (mod n) of the strand permutation π. Here $\pi = (2\,4\,6)(1\,5\,7\,3)$, with increments 2, 2, 3, 4, 2, 3, 5.

Hint: Consider the *necklace* $\mathcal{N}_\pi$ of boundary labels.

Geometry: Grassmannians and positroids

The Grassmannian $\mathrm{Gr}_{k,n}$ of (co)dimension k subspaces of $\mathbb{C}^n$ has homogeneous coordinate ring

$$\mathbb{C}[\mathrm{Gr}_{k,n}] = \mathbb{C}[\mathrm{Mat}_{k,n}]^{\mathrm{SL}_k},$$

i.e. SL_k-invariant functions of $k \times n$ matrices,

generated by Plücker coordinates (minors) $\Delta_J : J \in \binom{[n]}{k}$, satisfying (quadratic) Plücker relations.

The non-negative (real) Grassmannian

$$\mathrm{Gr}_{k,n}^{(\geqslant 0)} = \{w \in \mathrm{Gr}_{k,n}(\mathbb{R}) : \Delta_J(w) \geqslant 0,\ \forall J\} = \bigcup_{\mathcal{P}} \mathrm{Gr}_{\mathcal{P}}^{(>0)}$$

has a stratification indexed by *positroids* $\mathcal{P} \subseteq \binom{[n]}{k}$, where

$$\mathcal{P}(w) = \{J : \Delta_J(w) > 0\},\ \text{for } w \in \mathrm{Gr}_{k,n}^{(\geqslant 0)}.$$

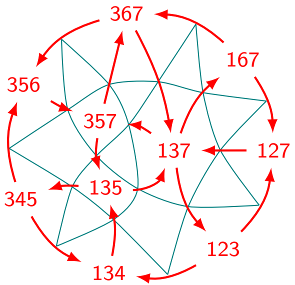
Clusters for positroid strata

A *cluster* $\mathcal{C} \subset \mathcal{P}$ is a subset such that

$$\mathrm{Gr}_{\mathcal{P}}^{(>0)} \to (\mathbb{R}_{>0})^{\mathcal{C}} \colon w \mapsto (\Delta_J(w))_{J \in \mathcal{C}}$$

is a bijection (i.e. a chart).

[Postnikov] Clusters $\mathcal{C} \subset \mathcal{P}_\pi$ are left source labels $\{\mathrm{J}_i : i \in Q_0\}$ for some dimer model on a disc with strand permutation π.



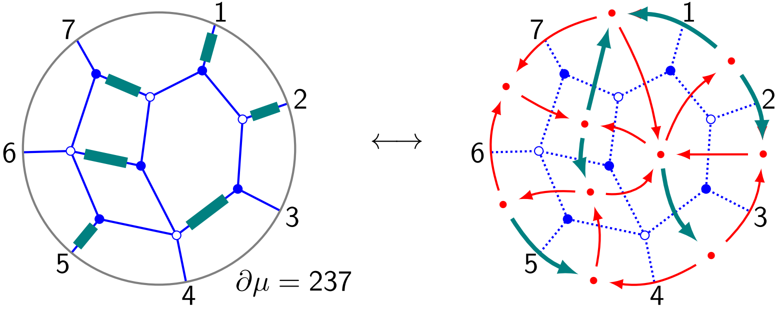
[Postnikov, Mu-Sp] The positroid $\mathcal{P}_\pi$ is the set of *boundary values of perfect matchings* for any dimer model with same strand permutation π. The necklace $\mathcal{N}_\pi$ is 'minimal' in the positroid $\mathcal{P}_\pi$.

Uniform (Grassmannian) case: if $\pi(i) = i + k$, then $\mathcal{P}_\pi = \binom{[n]}{k}$.

Perfect matchings on a quiver with faces

On G, a *perfect matching* μ is a collection of edges such that every node is incident with one edge in μ. Boundary value $\partial\mu$ is

$$\{\text{labels matched to a white node or unmatched to a black node}\}$$

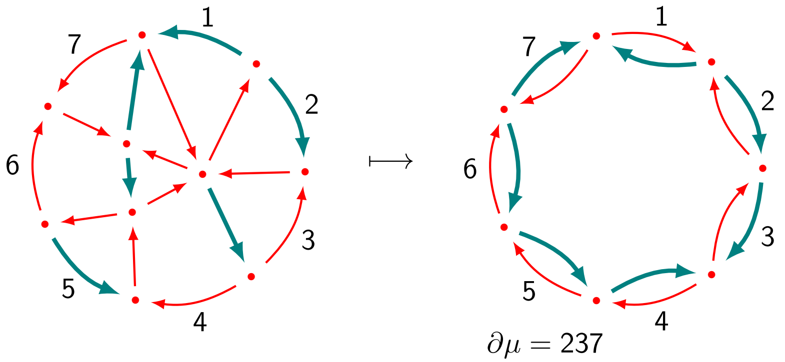


On Q, a *perfect matching* μ is a collection of arrows such that every face contains one arrow in μ.

Boundary values of matchings on Q

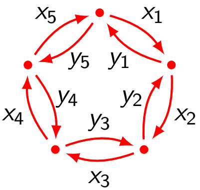
Boundary value $\partial\mu$ is the restriction of μ to the string of digons corresponding to boundary faces of Q. Identify $\partial\mu$ by $J \in \binom{[n]}{k}$ giving matched c-wise arrows. [*Ex:* it's the same k.]

$$k = \#\{\text{c-w faces}\} - \#\{\text{ac-w faces}\} + \#\{\text{ac-w bdry arrows}\}$$



Categorification I: circle algebra [JKS]

Define the *circle algebra* $C = C_{k,n}$ as the (complete) path algebra of a circle of n digons, modulo relations $xy = yx$, $y^k = x^{n-k}$.



Centre $Z = \mathbb{C}[[t]]$ for $t = xy$ and C is a Z-order, in particular, C is free and fin. gen. over Z.

The category CM C of C-modules free and fin. gen. over Z is a Frobenius cluster category, in particular, stably 2-Calabi-Yau.

CM C contains 'rank 1' modules M_J, for all $J \in \binom{[n]}{k}$, with

- Z at each vertex,
- (x_j, y_j) acting by $(t, 1)$, for $j \in J$, or $(1, t)$, for $j \notin J$.

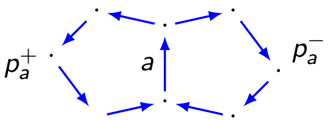
Thus M_J are 'rank 1 matrix factorisations of t' on each digon.

$\exists$ cluster character $\Psi\colon$ CM $C_{k,n} \to \mathbb{C}[\mathrm{Gr}_{k,n}]$ with $\Psi(M_J) = \Delta_J$.

Categorification II: dimer algebra [BKM]

For a dimer model Q on a disc, the *dimer algebra* $A = A_Q$ is the complete path algebra $\widehat{\mathbb{C}Q}$ modulo relations

$$p_a^+ = p_a^-$$



for each internal arrow $a \in Q_1^{int}$. (Alt: it's a frozen Jacobi algebra).

A central element t acts, at any vertex, as the boundary path of any adjacent face (all equal by relations). Thus A is also a Z-order.

Prop [CKP] Consistency implies that $e_j A e_i = \langle \text{paths } i \text{ to } j \rangle \cong Z$, so projectives Ae_i are rank 1, for all idempotents e_i, $i \in Q_0$.

Categorification III: boundary algebra

Define *boundary algebra* $B = eAe$ from idempotent $e = \sum_{bdry\ i} e_i$.
Note: $B = C$ in uniform (Grassmannian) case.

Thm [CKP] B naturally has C as a subalgebra and the restriction $\mathrm{CM}\, B \to \mathrm{CM}\, C$ is a fully faithful embedding, that is,

CM B modules are a special class of CM C modules.

Thm [Pres] $eA = \bigoplus_{i \in Q_0} eAe_i$ is a cluster tilting object in GP B (Gorenstein projective B-modules), which is a Frobenius cluster category contained in CM B. Also $A = \mathrm{End}_B(eA)^{\mathrm{op}}$.

Prop [CKP] M_J is in CM B if and only if $J \in \mathcal{P}$, the positroid.

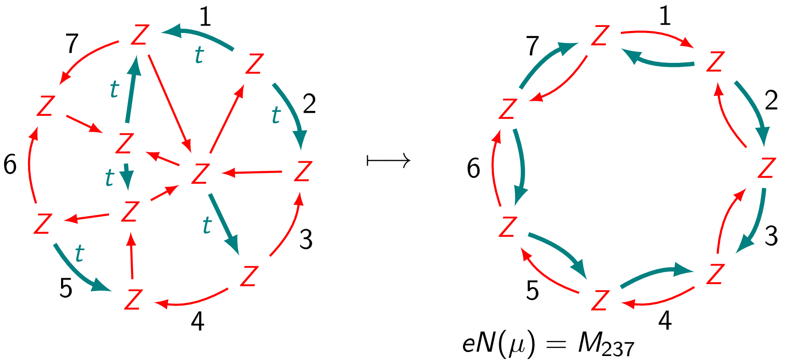
Proof by induction/restriction, i.e. $\mathrm{CM}\, A \to \mathrm{CM}\, B \colon N \mapsto eN$ and its adjoint $F \colon \mathrm{CM}\, B \to \mathrm{CM}\, A \colon M \mapsto \mathrm{Hom}_B(eA, M)$, we see that

rank 1 $M \in \mathrm{CM}\, B$ are the restrictions of rank 1 $N \in \mathrm{CM}\, A$.

Categorification IV: matching modules [CKP]

Each matching μ gives a rank 1 module $N(\mu) \in \mathrm{CM}\, A$ with Z at each vertex and arrows a acting by t, if $a \in \mu$, or 1, if $a \notin \mu$.

Thm All rank 1 modules have this form. Note: $eN(\mu) \cong M_{\partial\mu}$.



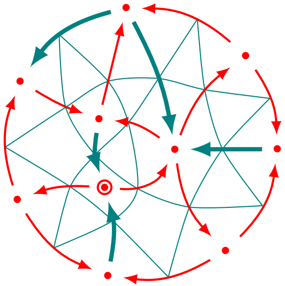
Thus $N(\mu)$ is a 'rank 1 matrix factorisation of t' on each face of Q.

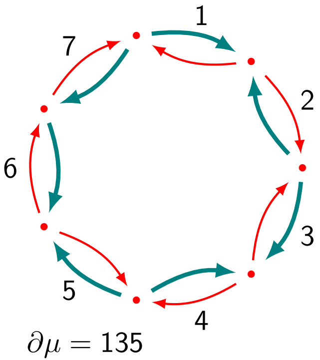
Projective matchings

Question: what are the boundary modules of the indecomposable projectives, i.e. the summands eAe_i of eA, for $i \in Q_0$?

Projectives Ae_i are rank 1, so $Ae_i \cong N(\mathtt{p}_i)$, for some matching $\mathtt{p}_i$:

$$a \in \mathtt{p}_i \iff \text{no minimal path from } i \text{ to } ha \text{ goes via } ta$$

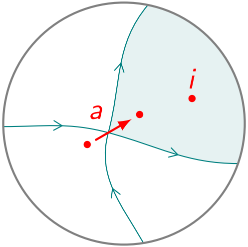




Muller-Speyer's downstream matchings

[Mu-Sp] defined canonical matchings $\mathtt{m}_i$, with $\partial\,\mathtt{m}_i = \mathtt{J}_i$, the left source label, by

$$a \in \mathtt{m}_i \iff i \in \text{d'stream wedge of } a.$$



Thm [CKP] $\mathtt{m}_i = \mathtt{p}_i$.

Cor 1 $eAe_i = M_{\mathtt{J}_i}$, so $eA = \bigoplus_{i \in Q_0} M_{\mathtt{J}_i}$

Cor 2 $B = eAe = \bigoplus_{J \in \mathcal{N}_\pi} M_J$, that is

the necklace $\mathcal{N}_\pi$ is the boundary algebra B (in CM C), whose rank 1 modules are the positroid $\mathcal{P}_\pi$.

Proof via projective resolution

Thm [CKP] Each matching module $N(\mu)$ in CM A has a projective resolution

$$\bigoplus_{\substack{a\in\mu \\ int}} Ae_{ta} \to \bigoplus_{a\notin\mu} Ae_{ha} \to \bigoplus_{i\in Q_0} Ae_i \to N(\mu),$$

so we can compute the class $[N(\mu)] \in \mathrm{K}(\mathrm{Proj}\, A)$.

Thm [CKP] $[N(\mathtt{m}_i)] = [Ae_i]$ and thus (with work) $N(\mathtt{m}_i) \cong Ae_i$.
Note: [Mu-Sp] knows this combinatorially, but with indirect proof.

Question: is there a direct combinatorial proof that $\mathtt{m}_i = \mathtt{p}_i$, i.e.
$i \notin$ d'stream wedge of $a \iff$ a min. path $i \to ha$ goes via ta ?

Matchings as cochains

For any quiver with faces Q, a *perfect matching* can be viewed as a function $\mu \in \mathbb{N}^{Q_1}$ s.t $d\mu = 1$, on all faces $f \in Q_2$ and thus an element of the lattice $\mathbb{M} = \{\mu \in \mathbb{Z}^{Q_1} : d\mu \in c(\mathbb{Z})\}$.

$$
\begin{array}{ccccccc}
\mathbb{Z} & \xrightarrow{\ c\ } & \mathbb{Z}^{Q_0} & \xrightarrow{\ d\ } & \mathbb{Z}^{Q_1} & \xrightarrow{\ d\ } & \mathbb{Z}^{Q_2} \\
\| & & \| & & \uparrow i & & \uparrow c \\
\mathbb{Z} & \xrightarrow{\ c\ } & \mathbb{Z}^{Q_0} & \xrightarrow{\ d\ } & \mathbb{M} & \xrightarrow{\ \deg\ } & \mathbb{Z}
\end{array}
$$

Top row is the reduced cochain complex: d is coboundary map, c is constant map, i is inclusion, deg is restriction of d.

Since Q is on a disc, both sequences are exact and so $\operatorname{rk} \mathbb{M} = |Q_0|$ and, in fact, $\{\mathtt{m}_i : i \in Q_0\}$ is a basis [Mu-Sp].

Partition functions and the network torus

The *partition functions* of the dimer model are

$$\mathcal{Z}_J = \sum_{\mu : \partial\mu = J} z^{\mu}, \quad \text{for each boundary value } J \in \binom{[n]}{k},$$

considered as formal Laurent polynomials (i.e. $z^m z^n = z^{m+n}$) or functions on the torus $\mathbb{T}$ whose character lattice is $\mathbb{M}$.
[Can think $\mathbb{T}$ is parametrised by 'edge weights modulo gauge'.]

Thm [Post'v,Mu-Sp] The map $\mathsf{n} \colon \mathbb{T} \to \widehat{\mathsf{Gr}}_{k,n}$ given algebraically by

$$\mathsf{n}^* \colon \mathbb{C}\big[\mathsf{Gr}_{k,n}\big] \to \mathbb{C}\big[\mathbb{T}\big] \colon \Delta_J \mapsto \mathcal{Z}_J$$

parametrises an open torus inside the associated positroid variety.

Categorified partition function

Regarding a module $N \in \mathrm{CM}\,A$ as a quiver representation (V_i, ϕ_a), there is a natural invariant $\nu\colon \mathrm{K}(\mathrm{CM}\,A) \to \mathbb{M}\colon [N] \mapsto \nu_N$ given by $\nu_N(a) = \dim \operatorname{coker} \phi_a$ and such that $\deg \nu_N = \operatorname{rk} N \; (= \operatorname{rk}_Z V_i)$.

For $M \in \mathrm{CM}\,B$, define the *module partition function*, generalising $\mathcal{Z}_J = \mathcal{Z}_{M_J}$,

$$\mathcal{Z}_M = \sum_{\substack{N \leqslant FM \\ eN = M}} z^{\nu_N},$$

where $F\colon M \mapsto \operatorname{Hom}_B(eA, M)$ is adjoint to restriction $N \mapsto eN$.
Note: families of N with fixed ν_N are counted by Euler char.

Conj/Thm [JKS3, in progress] $\mathcal{Z}_M = \mathrm{n}^* \Psi_M$

Some references

[BKM] K. Baur, A. King, B.R. Marsh, *Dimer models and cluster categories of Grassmannians*, Proc. Lond. Math. Soc. **113** (2016) 213–260 (arXiv:1309.6524)

[CKP] I. Canakci, A. King, M. Pressland, *Perfect matching modules, ...*, arXiv:2106.15924

[JKS] B.T. Jensen, A. King, X. Su, *A categorification of Grassmannian cluster algebras*, Proc. L.M.S. **113** (2016) 185–212 (arXiv:1309.7301)

[Mu-Sp] G. Muller, D.E. Speyer, *The twist for positroid varieties*, Proc. L.M.S. **115** (2017) 1014–1071 (arXiv:1606.08383)

[Pres] M. Pressland, *Calabi–Yau properties of Postnikov diagrams*, arXiv:1912.12475

[Scott] J. Scott, *Grassmannians and cluster algebras*, Proc. Lond. Math. Soc. **92** (2006) 345–380 (arXiv:math/0311148)